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Graded differential Lie algebras and model building

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Abstract

We develop a mathematical concept towards gauge field theories based upon a Hilbert space endowed with a representation of a skew-adjoint Lie algebra and an action of a generalized Dirac operator. This concept shares common features with the non-commutative geometry à la Connes/Lott, differs from that, however, by the implementation of skew-adjoint Lie algebras instead of unital associative $*$-algebras. We present the physical motivation for our approach and sketch its mathematical strategy. Moreover, we comment on the application of our method to the standard model and the flipped $SU(5)\times U(1)$-grand unification model. © 1998 Elsevier Science B.V.

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1. Physical motivation

We would like to construct (the classical action of) gauge field theories on a space–time manifold X with trivial topology out of the following input data:

(1) A unitary matrix Lie group G and its associated gauge group $\mathcal{G} = C^\infty(X) \otimes G$. Here, $C^\infty(X)$ denotes the algebra of real-valued smooth functions on X.
(2) Chiral fermions ψ transforming under a representation $\tilde{\pi}_0$ of G. The induced representation of the gauge group $\mathcal{G}$ is $\tilde{\pi} = \mathrm{id} \otimes \tilde{\pi}_0$.
(3) The fermionic mass matrix $\widetilde{\mathcal{M}}$, i.e. fermion masses plus generalized Kobayashi–Maskawa matrices.
(4) Possibly the spontaneous symmetry breaking pattern of G.

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Let us comment on these data. It is common knowledge that the free Dirac action for fermions,

$$S_{\mathrm{F}}^{\mathrm{free}}=\int_X \mathrm{d}x\, \psi^*(\mathsf{D}+\widetilde{\mathcal{M}})\psi, \tag{1}$$

is not gauge invariant. In this equation, D is the free Dirac operator and $\mathrm{d}x$ the volume form on X. First, the kinetic term $\psi^*\mathsf{D}\psi$ of the Dirac Lagrangian is not gauge invariant, because $\tilde{\pi}(\mathcal{G})$ does not commute with D. Usually, one restores gauge invariance by adding gauge fields A minimally coupled to the fermions. The gauge field A and its action on ψ are determined by the condition that there exist transformations of A under $\mathcal{G}$ that compensate the disturbing part of the transformation of $\psi^*\mathsf{D}\psi$. Second, if the action of only a subgroup $\mathcal{G}_0$ of $\mathcal{G}$ commutes with $\widetilde{\mathcal{M}}$, then the mass term $\psi^*\widetilde{\mathcal{M}}\psi$ of the Dirac Lagrangian is not gauge invariant. In this case, one restores gauge invariance by extending the fermionic mass matrix to Higgs fields $\widetilde{\mathcal{M}}+\Phi$ with appropriate transformation behavior. Thus, the gauge invariant fermionic action can be written symbolically (i.e. up to signs and constants of the order one) as

$$S_{\mathrm{F}}^{\mathrm{inv}}=\int_X \mathrm{d}x\, \psi^*(\mathsf{D}+\widetilde{\mathcal{M}}+\mathrm{A}+\Phi)\psi. \tag{2}$$

Moreover, one wishes to have a dynamics for the fields A and Φ. This is achieved by adding the free bosonic action

$$S_{\mathrm{B}}^{\mathrm{free}}=\int_X \mathrm{d}x(\langle \mathbf{d}\mathrm{A},\mathbf{d}\mathrm{A}\rangle_2+\langle \mathbf{d}(\Phi+\widetilde{\mathcal{M}}),\mathbf{d}(\Phi+\widetilde{\mathcal{M}})\rangle_1)\,, \tag{3}$$

where $\langle\ ,\ \rangle_2$ and $\langle\ ,\ \rangle_1$ are appropriate scalar products. However, the action $S_{\mathrm{B}}^{\mathrm{free}}$ is not gauge invariant, one has to add interaction terms for A and Φ. Moreover, the vacuum expectation value of $\Phi+\widetilde{\mathcal{M}}$ must be just the mass matrix $\widetilde{\mathcal{M}}$ in order to reproduce the correct fermionic sector. This is achieved by adding quartic interaction terms $V(\Phi+\widetilde{\mathcal{M}})$ such that $\Phi+\widetilde{\mathcal{M}}=\widetilde{\mathcal{M}}$ is a local minimum of $V(\Phi+\widetilde{\mathcal{M}})$. Here, one has to implement the desired spontaneous symmetry breaking scheme (4), which in some gauge theories is already determined by the fermionic mass matrix $\widetilde{\mathcal{M}}$. However, in extended theories, one may need supplementary information on the spontaneous symmetry breaking scheme that is not contained in $\widetilde{\mathcal{M}}$. In summary, the invariant bosonic action has the symbolic form

$$\begin{aligned}S_{\mathrm{B}}^{\mathrm{inv}}=\int_X \mathrm{d}x(&\langle \mathbf{d}\mathrm{A}+\mathrm{A}^2,\mathbf{d}\mathrm{A}+\mathrm{A}^2\rangle_2\\&+\langle(\mathbf{d}+\mathrm{A})(\Phi+\widetilde{\mathcal{M}}),(\mathbf{d}+\mathrm{A})(\Phi+\widetilde{\mathcal{M}})\rangle_1+V(\Phi+\widetilde{\mathcal{M}})).\end{aligned} \tag{4}$$

We see that our input data (1)–(4) should suffice to reconstruct a complete classical gauge field theory. In particular, the fermionic sector determines candidates for the bosonic configuration space. Of course, the actions (2) and (4) are not unique, but we can fix much of the ambiguity by a minimal choice of A and Φ.

Usually, the above construction scheme is carried out more or less by hand. This is not difficult, for example, in the case of the standard model. However, in grand unified theories with very large Higgs multiplets this is a highly non-trivial puzzle. One may wish to have a machinery at disposal which is able to do this work. This machinery should consist of an algorithm which has to be fed with the data (1)–(4) as the input and which returns the desired action, in particular, the Higgs multiplets and the Higgs potential. This paper is a sketch of such a machinery, which even does much more: It also returns tree-level predictions for the masses of Yang–Mills and Higgs fields.

An idea how to find this machinery is inspired by the following observation [23]: The gauge field A is a vector field and the Higgs field Φ a scalar field. From that point of view, both are completely different objects. However, in the above sketch they play precisely the same role. Both A and Φ occur via minimal coupling in the fermionic action (2) and restore in this way the gauge invariance. Both have the same type of kinetic Lagrangians (3). Both occur as fourth-order polynomials in the bosonic action (4). Moreover, also D and $\widetilde{\mathcal{M}}$ play the same role. All that may be an accident. But accidents have often inspired new theories. It might be promising to search for a new type of mathematics that deals with vector and scalar fields in the same way. Such mathematics does already exist in form of Alain Connes' non-commutative geometry [7]!

2. Non-commutative geometry

2.1. General remarks

The evolution of non-commutative topology started with Gel'fands discovery that the unital C^*-algebra $C(X)$ of continuous functions over a compact manifold X contains all information about that manifold: Given $C(X)$ one can reconstruct the manifold X (up to homeomorphisms) as the set of characters. In the other direction, each commutative unital C^*-algebra is isomorphic to $C(X)$ for a certain compact manifold X. This language was transcribed to the case that the C^*-algebra is not commutative, and one considers general C^*-algebras as function algebras over "non-commutative manifolds". This programme, to dualize geometric or topological objects and to deform them within the dual picture, has been very successful. It led for instance to algebraic K-theory and quantum groups.

2.2. The Connes–Lott prescription

Gel'fands theorem establishes the duality between the function algebra $C(X)$ and the topology of X. The discovery of Connes [7,9] was that, taking in addition the Dirac operator acting on the spinor Hilbert space, one can also recover the metric properties of X. It is possible to reconstruct the distance between two points and the de Rham complex. Formalizing this method, Connes introduced the basic object of non-commutative geometry, the K-cycle or [2] spectral triple:

[2] We prefer the ancient notation 'K-cycle'.

Definition 1. A K-cycle $(\mathcal{A}, h, D, \pi, \Gamma)$ over a unital associative $*$-algebra $\mathcal{A}$ is given by:
(i) an involutive representation π of $\mathcal{A}$ in the algebra $\mathcal{B}(h)$ of bounded operators on a Hilbert space h,
(ii) a (possibly unbounded) self-adjoint operator D on h such that $(1_{\mathcal{B}(h)} + D^2)^{-1}$ is compact and for all $a \in \mathcal{A}$ there is $[D, \pi(a)] \in \mathcal{B}(h)$.
The K-cycle is called even iff in addition there is a self-adjoint operator Γ on h, fulfilling $\Gamma^2 = 1_{\mathcal{B}(h)}$, $\Gamma D + D\Gamma = 0$ and $\Gamma\pi(a) - \pi(a)\Gamma = 0$, for all $a \in \mathcal{A}$.

Non-commutative geometry (NCG) as sketched above seems to be perfectly adapted to the setting (1)–(4): For technical reasons one first has to pass from the space–time manifold to a compact Euclidian spin manifold X. Then the fermions ψ constitute the Hilbert space h. Next, one chooses the self-adjoint operator D of Definition 1 to be equal to $\mathrm{D} + \widetilde{\mathcal{M}}$ on physical fermions ψ. A matrix algebra $\mathcal{A}_M$ is chosen in such a way that the gauge group $\mathcal{G} = C^\infty(X) \otimes G$ is isomorphic to the group of unitary elements of the algebra $\mathcal{A} = C^\infty(X) \otimes \mathcal{A}_M$. The action $\pi = \mathrm{id} \otimes \pi_0$ of $\mathcal{A} = C^\infty(X) \otimes \mathcal{A}_M$ on h is the extension [3] of the group representation $\tilde{\pi} = \mathrm{id} \otimes \tilde{\pi}_0$ of $\mathcal{G} = C^\infty(X) \otimes G$ on the fermions ψ. At the very end, one returns to an indefinite metric by a Wick rotation. Chiral fermions are obtained by means of a chirality condition via the operator Γ.

To any K-cycle $(\mathcal{A}, h, D, \pi, \Gamma)$ there is canonically associated a differential algebra $\Omega_D^*\mathcal{A}$: One considers the universal graded differential algebra $\Omega^*\mathcal{A}$ over the algebra $\mathcal{A}$ of the K-cycle,

$$\Omega^*\mathcal{A} = \bigoplus_{n=0}^{\infty} \Omega^n\mathcal{A}, \qquad \Omega^n\mathcal{A} = \left\{ \sum_\alpha a_\alpha^0 \, \mathrm{d}a_\alpha^1 \, \mathrm{d}a_\alpha^2 \ldots \mathrm{d}a_\alpha^n \right\}, \tag{5}$$

where d is the universal differential and $a_\alpha^i \in \mathcal{A}$. In particular, $\Omega^0\mathcal{A} \cong \mathcal{A}$. One defines a linear representation π of $\Omega^*\mathcal{A}$ on the Hilbert space h by [25]

$$\pi(a_0 \, \mathrm{d}a_1 \, \mathrm{d}a_2 \ldots \mathrm{d}a_n) := \pi(a_0) \cdot [-\mathrm{i}D, \pi(a_1)] \cdot [-\mathrm{i}D, \pi(a_2)] \cdots [-\mathrm{i}D, \pi(a_n)]. \tag{6}$$

One remarks that $\pi(\Omega^*\mathcal{A})$ is not a differential algebra. Fortunately, this defect can be repaired, and the canonical graded differential algebra is

$$\begin{aligned} \Omega_D^n\mathcal{A} = \bigoplus_{n=0}^{\infty} \Omega_D^n\mathcal{A}, \quad \Omega_D^n\mathcal{A} &:= \Omega^n\mathcal{A}/((\ker\pi + \mathrm{d}\ker\pi) \cap \Omega^n\mathcal{A}) \\ &\cong \pi(\Omega^n\mathcal{A})/\pi(\mathrm{d}\ker\pi \cap \Omega^n\mathcal{A}). \end{aligned} \tag{7}$$

For the physically interesting case of even K-cycles over a subalgebra of $C^\infty(X)_{\mathbb{C}} \otimes \mathrm{M}_F\mathbb{C}$ and generalized Dirac operators of the form $D = \mathrm{D} \otimes 1_F + \gamma^5 \otimes \mathcal{M}$, a generally applicable construction of $\Omega_D^*\mathcal{A}$ has been given in [18]. The non-commutative gauge potential is an element of $\Omega_D^1\mathcal{A}$ and the field strength an element of $\Omega_D^2\mathcal{A}$. Using invariant scalar products

[3] Provided that this is possible!

one defines bosonic and fermionic actions [7,9]. A further improvement is a new spectral action principle [2,3] that gives a coupling of the Yang–Mills (–Higgs) action to Einstein plus Weyl gravity.

2.3. Application to the standard model

This NCG-prescription has proved very successful in reformulating the standard model. There exists an "old scheme" initiated by Connes and Lott [9], see also [7,16,19,20,25], and a "new scheme" based upon real structures introduced by Connes [8], see also [1,17,23] for the application to model building. The algebra $\mathcal{A}$ and its group of unitary elements $\mathcal{U}(\mathcal{A})$ are given by

$$\begin{aligned} \mathcal{A}_{\text{old}} &= C^\infty(X) \otimes ((\mathbb{H} \oplus \mathbb{C}) \oplus (\mathrm{M}_3\mathbb{C} \oplus \mathbb{C})), \\ \mathcal{U}(\mathcal{A}_{\text{old}}) &= C^\infty(X) \otimes (\mathrm{SU}(2) \times \mathrm{U}(1) \times \mathrm{U}(3) \times \mathrm{U}(1)), \\ \mathcal{A}_{\text{new}} &= C^\infty(X) \otimes (\mathbb{H} \oplus \mathbb{C} \oplus \mathrm{M}_3\mathbb{C}), \\ \mathcal{U}(\mathcal{A}_{\text{new}}) &= C^\infty(X) \otimes (\mathrm{SU}(2) \times \mathrm{U}(1) \times \mathrm{U}(3)). \end{aligned} \tag{8}$$

The additional U(1)-groups are eliminated by unimodularity conditions. The most important improvement compared with the usual formulation of the standard model is that the non-commutative gauge potential contains both the $\mathrm{su}(3) \oplus \mathrm{su}(2) \oplus \mathrm{u}(1)$ Yang–Mills fields and the complex Higgs doublet. Moreover, the bosonic action contains the Yang–Mills Lagrangian, the covariant derivatives of the Higgs fields and the Higgs potential in a unified form. The fermionic action unifies the gauge field couplings with the Yukawa-couplings. Numerically, one gets a very promising "fuzzy" relation between the mass of the W boson and the mass of the top quark, and the prediction for the mass of the Higgs field is compatible with LEP precision experiments, see [1,7,8,16,17,19,23].

2.4. The Mainz–Marseille model

There exists a different NCG-formulation of the standard model [10–12,14] elaborated by groups in Mainz and Marseille. This formulation leads to the same unification of the Yang–Mills and Higgs sectors in the bosonic and fermionic actions. The essential mathematical difference is the use of the graded Lie algebra $\Lambda^* \otimes \mathrm{su}(2|1)$ of differential form-valued matrices as the starting point instead of K-cycles and differential algebras constructed thereof in the Connes–Lott prescription. The essential physical difference is that the purely bosonic sector of the standard model can be formulated. This is in contrast to the Connes–Lott model, where the bosonic sector can only be reproduced if at least two generations of fermions occur in nature (which is the case, of course). The Mainz–Marseille model yields no relations between fermion and boson masses, but an interesting relation between the Cabibbo angle and quark masses can be obtained [10,15].

The inseparable tie between bosons and fermions in the Connes–Lott model, which is responsible for relations between fermion and boson masses obtained in that model, has been criticized by the Mainz–Marseille group, mainly for two reasons: First, purely bosonic theories are mathematically interesting as well. Second, relations between fermion and boson masses do not survive the usual quantization procedure. However, there exist examples where parameter relations that are not stemming from a symmetry of the theory are respected on quantum level [29]. Thus, our point of view is to consider the interpretation of the mass relations in the Connes–Lott model as a challenge for the future.

2.5. *Non-commutative geometry and grand unification*

The overwhelming success of non-commutative geometry leads to the expectation that its application to other gauge field theories should not be difficult. However, if one follows the Connes–Lott prescription, one runs into certain problems. It was shown in [22] that, besides the standard model, there are only two more or less realistic models which can be constructed within the above understanding of NCG: the $SU(4)_{PS} \times SU(2)_L \times SU(2)_R$-model and the $SU(3)_C \times SU(2)_L \times SU(2)_R \times U(1)_{B-L}$-model. However, if one additionally demands a real structure [8] for the K-cycle, then also these two models are ruled out. The only more or less realistic physical model that is compatible with the most elegant NCG-prescription is the standard model! It is certainly too early to judge from experimental results whether the standard model is correct or not. At least there exist good reasons [21] why one could be interested in grand unified theories (GUTs): GUTs explain the quantization of electric charge, yield a fairly good prediction for the Weinberg angle, explain the convergence of running coupling constants at high energies, include massive neutrinos to solve the solar neutrino problem, produce the observed baryon asymmetry of the universe, etc. Unfortunately, the results of [22] imply that one needs additional structures or different methods for a NCG-formulation of these models.

Perhaps, the most successful NCG-approach towards grand unification was proposed by Chamseddine et al. [4,5]. In the SU(5)-model [4,5], the authors start to construct an auxiliary K-cycle. Within this framework they construct the bosonic sector. Then they interpret some of these bosonic quantities as Lie algebra valued and consider Lie algebra representations on the physical Hilbert space to obtain the fermionic sector. This procedure is a systematic realization of the gauge theory construction programme set up at the beginning. However, an aesthetic shortcoming of that approach is the auxiliary character of the K-cycle, which of course is inevitable in view of [22]. The SO(10)-model by Chamseddine and Fröhlich [6] fits well [4] into the NCG-scheme. The reason why this model was excluded in [22] is that only models possessing complex fundamental irreducible representations were admitted in that article.

It turns out that only a slight modification of the Connes–Lott prescription enables the formulation of a large class of physical models without additional structures. A sketch of that formulation and of its application to interesting physical models is the concern of this paper.

[4] Nevertheless, the use of Lie algebras instead of algebras could probably justify certain assumptions made in [6].

3. A modification of non-commutative geometry

Let us investigate why the most elegant NCG-prescription is so restrictive to admissible models. The obstruction is the extension of the representations of the gauge group $\mathcal{G} = C^\infty(X) \otimes G$ to representations of the unital associative $*$-algebra $\mathcal{A} = C^\infty(X) \otimes \mathcal{A}_M$ containing $\mathcal{G}$ as the set of unitary elements. That $\tilde{\pi} = \mathrm{id} \otimes \tilde{\pi}_0$ is a representation of $\mathcal{G}$ on the Hilbert space h means that

$$\tilde{\pi}_0(g_1)\,\tilde{\pi}_0(g_2) = \tilde{\pi}_0(g_1 g_2) \quad \forall g_1, g_2 \in G. \tag{9}$$

The representation $\tilde{\pi}_0$ of the matrix group G should coincide with the representation π_0 of the matrix algebra $\mathcal{A}_M$ on the subset $G \subset \mathcal{A}_M$,

$$\pi_0(g_1)\pi_0(g_2) = \pi_0(g_1 g_2) \quad \forall g_1, g_2 \in G \subset \mathcal{A}_M. \tag{10}$$

It is, perhaps, not the problem to extend the multiplication rule (10) to the entire matrix algebra $\mathcal{A}_M$. The essential problem is that this extension must be compatible with linear operations,

$$\lambda_1 \pi_0(a_1) + \lambda_2 \pi_0(a_2) = \pi_0(\lambda_1 a_1 + \lambda_2 a_2) \quad \forall a_1, a_2 \in \mathcal{A}_M, \quad \forall \lambda_1, \lambda_2 \in \mathbb{R}. \tag{11}$$

Addition and multiplication by scalars are not defined on G , and the representation $\tilde{\pi}_0$ does not care whether it is linear or not. A priory, there are two types of irreducible representations that fulfil (11): the identity and – in the case of real algebras – the complex conjugation. In general, this is all what is possible. We see: The reason why the most elegant NCG-prescription [8] is so restrictive is that it is compatible only with linear representations of the matrix group. Most of the GUT are not of that type.

Fortunately, our observation also shows the way how to overcome the restriction: We propose to linearize the matrix group, which means to work within the tangent space at a fixed group element, for instance the unit element. The tangent space at the unit element is isomorphic to the Lie algebra $\mathfrak{a}$ of G. Thus, *the Lie algebra $\mathfrak{g} = C^\infty(X) \otimes \mathfrak{a}$ of the gauge group $\mathcal{G} = C^\infty(X) \otimes G$ is the correct object to use, not an algebra extending $\mathcal{G}$.* The linearized group multiplication is described by the commutator of Lie algebra elements. It is clear that the representation of a Lie group induces a representation of its Lie algebra. The point is that this Lie algebra representation is always linear.

In analogy to the procedure in non-commutative geometry we formalize our observation. We simply replace in Definition 1 the unital associative $*$-algebra $\mathcal{A}$ by a skew-adjoint Lie algebra $\mathfrak{g}$. The outcome can no longer be called a K-cycle; I propose the name "L-cycle", where the letter L stands for Lie (and it is the next letter in the alphabet).

Definition 2. An L-cycle $(\mathfrak{g}, h, D, \pi, \Gamma)$ over a skew-adjoint Lie algebra $\mathfrak{g}$ is given by

(i) an involutive representation π of $\mathfrak{g}$ in the Lie algebra $\mathcal{B}(h)$ of bounded operators on a Hilbert space h, i.e. $(\pi(a))^* = \pi(a^*) \equiv -\pi(a)$, for any $a \in \mathfrak{g}$,

(ii) a (possibly unbounded) self-adjoint operator D on h such that $(\mathrm{id}_h + D^2)^{-1}$ is compact and for all $a \in \mathfrak{g}$ there is $[D, \pi(a)] \in \mathcal{B}(h)$, where id_h denotes the identity on h.

(iii) a self-adjoint operator Γ on h, fulfilling $\Gamma^2 = \mathrm{id}_h$, $\Gamma D + D\Gamma = 0$ and $\Gamma\pi(a) - \pi(a)\Gamma = 0$, for all $a \in \mathfrak{g}$.

It seems obvious that our concept is perfectly adapted to the setting (1)–(4) at the beginning: [5] In the same way as in the Connes–Lott formulation we start with the construction of a Euclidian gauge field theory. Again, the Euclidian fermions ψ constitute our Hilbert space h. For technical reasons it may sometimes be necessary to work with several copies of the fermions. The Lie algebra $\mathfrak{g} = C^\infty(X) \otimes \mathfrak{a}$ is simply the Lie algebra of the gauge group $\mathcal{G}$, up to possible modifications if U(1)-groups occur (see footnote 5). We assume that X has a trivial topology in order to avoid discussions of transition functions between different charts of the manifold. The Lie algebra representation $\pi = \mathrm{id} \otimes \hat{\pi}$ is just the differential $\tilde{\pi}_*$ of the group representation $\tilde{\pi} = \mathrm{id} \otimes \tilde{\pi}_0$. The self-adjoint operator D is chosen in such a way that on physical fermions it equals $\mathsf{D} + \widetilde{\mathcal{M}}$. The operator Γ represents the chirality properties of the fermions. Finally, one returns to Minkowski space by a Wick rotation and imposes a chirality condition for the fermions ψ by means of Γ.

The programme of our approach is clear: We "simply" have to transcribe the Connes–Lott prescription of non-commutative geometry to our case. However, this is not as easy as one probably expects. The associativity of the algebra and the existence of a unit element are very powerful tools. Without them we are forced to go long detours where non-commutative geometry uses short cuts.

4. The general scheme

Now for the sketch of the construction in the general context, without relation to physical models, a detailed exposition of our techniques can be found in [26]. In analogy to the first step in the Connes–Lott prescription we enlarge our Lie algebra $\mathfrak{g}$ to a universal graded differential Lie algebra $\Omega^*\mathfrak{g}$. One can imagine $\Omega^*\mathfrak{g}$ as the set of repeated graded commutators of $\mathfrak{g}$ and $\mathrm{d}\mathfrak{g}$, where $\mathrm{d}\mathfrak{g}$ is a second copy of $\mathfrak{g}$. Thus, elements $\omega \in \Omega^*\mathfrak{g}$ have the form

$$\omega = \sum_{\alpha, z \geq 0} [v_\alpha^z, [v_\alpha^{z-1}, [\ldots, [v_\alpha^1, v_\alpha^0] \ldots]]], \quad \text{finite sum}, \tag{12}$$

where v_α^i either belongs to $\mathfrak{g}$ or $\mathrm{d}\mathfrak{g}$. The vector space $\Omega^*\mathfrak{g}$ is $\mathbb{N}$-graded. The homogeneous element $[v^z, [v^{z-1}, [\ldots, [v^1, v^0] \ldots]]]$ belongs to $\Omega^n\mathfrak{g}$ iff n elements of $\{v^0, \ldots, v^z\}$ belong to $\mathrm{d}\mathfrak{g}$. The graded commutator $[\ ,\ ]$ is compatible with that grading structure; one has $[\Omega^k\mathfrak{g}, \Omega^l\mathfrak{g}] \subset \Omega^{k+l}\mathfrak{g}$. Moreover, $[\ ,\ ]$ respects the usual graded antisymmetry and the graded Jacobi identity. The symbol d is extended to a graded differential on $\Omega^*\mathfrak{g}$, it is nilpotent and obeys the graded Leibniz rule. The graded Lie algebra $\Omega^*\mathfrak{g}$ is universal in the following sense: Each graded differential Lie algebra generated by $\pi(\mathfrak{g})$ and $\mathrm{d}\pi(\mathfrak{g})$ can

[5] There can occur obstructions and modifications if Abelian Lie groups are present. In particular, a purely Abelian gauge field theory can be constructed only with partial success. In some cases, Abelian Lie algebras are automatically generated. If such a Lie algebra is desired, one can omit this part when deriving the Lie algebra $\mathfrak{g}$ out of $\mathcal{G}$.

be obtained by factorization of $\Omega^*\mathfrak{g}$ with respect to a differential ideal. For instance, the information contained in an L-cycle determines uniquely such a differential ideal. Thus, there is a canonical graded differential Lie algebra $\Omega_D^*\mathfrak{g}$ associated to an L-cycle.

To find this differential Lie algebra, we represent $\Omega^*\mathfrak{g}$ on the Hilbert space h, using the data specified in the L-cycle. This representation extends the representation π of the L-cycle and is defined by

$$\begin{aligned} \pi(\mathrm{d}a) &= [-\mathrm{i}D, \pi(a)], \\ \pi([\omega^k, \tilde{\omega}^l]) &= [\pi(\omega^k), \pi(\tilde{\omega}^l)]_g := \pi(\omega^k)\pi(\tilde{\omega}^l) - (-1)^{kl}\pi(\tilde{\omega}^l)\pi(\omega^k), \end{aligned} \tag{13}$$

for $a \in \mathfrak{g}$, $\omega^k \in \Omega^k\mathfrak{g}$ and $\tilde{\omega}^l \in \Omega^l\mathfrak{g}$. Here, it is essential to have the grading operator Γ, which detects the correct sign for $(-1)^{kl}$.

As one expects from the Connes–Lott formulation, the representation π does not transport the differential d on $\Omega^*\mathfrak{g}$ to a differential on $\pi(\Omega^*\mathfrak{g})$. To cure this, we use the usual trick of non-commutative geometry. One shows that

$$\mathcal{J}^*\mathfrak{g} = \ker\pi + \mathrm{d}\ker\pi \subset \Omega^*\mathfrak{g} \tag{14}$$

is a graded differential ideal of $\Omega^*\mathfrak{g}$. Factorizing out the "junk" $\mathcal{J}^*\mathfrak{g}$ we obtain the graded differential Lie algebra $\Omega_D^*\mathfrak{g}$,

$$\Omega_D^*\mathfrak{g} = \bigoplus_{n=0}^{\infty} \Omega_D^n\mathfrak{g}, \qquad \Omega_D^n\mathfrak{g} = \frac{\Omega^n\mathfrak{g}}{\mathcal{J}^n\mathfrak{g}} \cong \frac{\pi(\Omega^n\mathfrak{g})}{\pi(\mathcal{J}^n\mathfrak{g})}. \tag{15}$$

The differential and the commutator are defined as usual for equivalence classes.

It is extremely useful to introduce a linear map σ from $\Omega^*\mathfrak{g}$ to (possibly unbounded) operators on h. The operator σ is odd with respect to the $\mathbb{Z}_2$-grading and is within the same notations as before defined by

$$\begin{aligned} &\sigma(a) = 0, \qquad \sigma(\mathrm{d}a) = [D^2, \pi(a)], \\ &\sigma([\omega^k, \tilde{\omega}^l]) = [\sigma(\omega^k), \pi(\tilde{\omega}^l)]_g + (-1)^k[\pi(\omega^k), \sigma(\tilde{\omega}^l)]_g. \end{aligned} \tag{16}$$

The importance of the map σ is that it measures the defect if one represents the universal differential d by graded commutators with $-\mathrm{i}D$,

$$\pi(\mathrm{d}\omega^k) = [-\mathrm{i}D, \pi(\omega^k)]_g + \sigma(\omega^k), \quad \omega^k \in \Omega^k\mathfrak{g}. \tag{17}$$

In particular, taking $\omega^k \in \ker\pi$, we get

$$\pi(\mathcal{J}^{k+1}\mathfrak{g}) = \{\sigma(\omega^k), \omega^k \in \Omega^k \cap \ker\pi\}. \tag{18}$$

This characterization of $\pi(\mathcal{J}^*\mathfrak{g})$ is especially convenient, because $\sigma(\omega^k)$ is derived successively from lower degrees, see (16). Indeed, this is the way how we can eventually compute $\pi(\mathcal{J}^*\mathfrak{g})$: The real problem is to find $\sigma(\Omega^1\mathfrak{g})$. Then we derive for $k \geq 2$ by induction a formula for $\sigma(\omega^k)$ for given $\pi(\omega^k)$. Clearly, $\sigma(\omega^k)$ is not uniquely defined by $\pi(\omega^k)$, and this ambiguity is nothing but $\pi(\mathcal{J}^{k+1}\mathfrak{g})$. However, the explicit realization of this line is not done within a couple of pages. We also point out that, once knowing $\sigma(\omega^k)$, formula (17) provides the explicit differentiation rule for elements of $\Omega_D^*\mathfrak{g}$.

In the Connes–Lott formulation of non-commutative geometry, all work is done at this point. There, the connection form is simply an element of $\Omega_D^1\mathcal{A}$ and the curvature an element of $\Omega_D^2\mathcal{A}$. It is straightforward to write down the fermionic and bosonic actions. In our case, the situation is different. If one tries to find a reasonable definition for the connection (the covariant derivative), one encounters more freedom than one expects. Moreover, it is not possible to describe gauge field theories containing U(1)-groups if one takes $\Omega_D^1\mathfrak{g}$-valued connection forms. Therefore, an additional structure is necessary: Not the graded differential Lie algebra $\Omega_D^*\mathfrak{g}$ is the correct space where the connection form and the curvature live, but the space of certain graded Lie endomorphisms of $\Omega_D^*\mathfrak{g}$. This is not completely unreasonable. For instance, connections within the framework of finite projective modules [24] are of a similar type. Formally, we introduce the space $\mathcal{H}^*\mathfrak{g} = \bigoplus_{n\in\mathbb{N}} \mathcal{H}^n\mathfrak{g}$ of certain graded Lie homomorphisms of $\pi(\Omega^*\mathfrak{g})$. The space $\mathcal{H}^n\mathfrak{g}$ consists of linear (possibly unbounded) operators on h of $\mathbb{Z}_2$-degree n mod 2, which raise the $\mathbb{N}$-degree of $\pi(\Omega^*\mathfrak{g})$ and $\pi(\mathcal{J}^*\mathfrak{g})$ by n,

$$[\mathcal{H}^n\mathfrak{g}, \pi(\Omega^k\mathfrak{g})]_g \subset \pi(\Omega^{k+n}\mathfrak{g}), \qquad [\mathcal{H}^n\mathfrak{g}, \pi(\mathcal{J}^k\mathfrak{g})]_g \subset \pi(\mathcal{J}^{k+n}\mathfrak{g}). \tag{19}$$

Factorizing $\mathcal{H}^*\mathfrak{g}$ with respect to its graded centralizer $\tilde{\mathbf{c}}^*\mathfrak{g}$ in $\pi(\Omega^*\mathfrak{g})$ and the ideal $\pi(\mathcal{J}^*\mathfrak{g})$, we obtain the graded Lie algebra

$$\hat{\mathcal{H}}^*\mathfrak{g} := \bigoplus_{n\in\mathbb{N}} \hat{\mathcal{H}}^n\mathfrak{g}, \quad \hat{\mathcal{H}}^n\mathfrak{g} := \mathcal{H}^n\mathfrak{g}/(\pi(\mathcal{J}^n\mathfrak{g} + \tilde{\mathbf{c}}^n\mathfrak{g}). \tag{20}$$

The differential and the commutator on $\hat{\mathcal{H}}^*\mathfrak{g}$ are defined as usual for dual spaces: via the graded Leibniz rule and the graded Jacobi identity. From our definitions it is clear that

$$\pi(\Omega^n\mathfrak{g}) \subset \mathcal{H}^n\mathfrak{g}, \quad \Omega_D^n\mathfrak{g} \subset \hat{\mathcal{H}}^n\mathfrak{g}. \tag{21}$$

In some sense, this framework is an extension of the primary spaces $\pi(\Omega^*\mathfrak{g})$ and $\Omega_D^*\mathfrak{g}$.

The formal definition of a connection on L-cycles is given in [26]. Here, we shall only quote the result: A connection ∇ acting on $\Omega_D^*\mathfrak{g}$ is closely related to the covariant derivative ∇_h acting on the Hilbert space h. The general form of these two objects is

$$\nabla_h = -\mathrm{i}D + \rho, \quad \nabla = d + [\tilde{\rho}, .]_g, \qquad \rho \in \mathcal{H}^1\mathfrak{g}, \;\; \tilde{\rho} := \rho + \tilde{\mathbf{c}}^1\mathfrak{g} \in \hat{\mathcal{H}}^1\mathfrak{g}. \tag{22}$$

The Lie homomorphism ρ is called the connection form (gauge potential). The curvature (field strength) of the connection ∇ is

$$\nabla^2 = [\theta, .], \quad \theta = \mathrm{d}\tilde{\rho} + \tfrac{1}{2}\{\tilde{\rho}, \tilde{\rho}\} \in \hat{\mathcal{H}}^2\mathfrak{g}. \tag{23}$$

We see that our formulae look very similar to what one knows from non-commutative geometry or classical gauge field theory. However, we have no control over the space of connections in that general context. All that we know is that elements of $\Omega_D^1\mathfrak{g}$ are possible connection forms, but it is completely unclear what else. Also the operations $\mathrm{d}\tilde{\rho}$ and $\{\tilde{\rho}, \tilde{\rho}\}$ are difficult to perform, because they are only indirectly defined. It is a visible complication compared with the Connes–Lott prescription to find not only $\Omega_D^*\mathfrak{g}$ but also $\hat{\mathcal{H}}^*\mathfrak{g}$ (up to second degree).

The group $\mathcal{U}(\mathfrak{g})$ obtained via the exponential mapping of a neighbourhood of the zero element of $\mathcal{H}^0\mathfrak{g}$ plays the role of a gauge group in our approach. Comparing, for a physical model, this group with the original gauge group $\mathcal{G}$ we had started with, we see that the global topology of $\mathcal{G}$ cannot always be reconstructed. But for most physical applications it suffices to know the gauge group locally. One can define an adjoint representation Ad of $\mathcal{U}(\mathfrak{g})$ on $\Omega^*_D\mathfrak{g}$. Local gauge transformations are given by

$$\begin{aligned} &\nabla \mapsto \mathrm{Ad}_u \nabla \mathrm{Ad}_{u^*}, && \nabla_h \mapsto u\nabla_h u^*, \\ &\rho \mapsto u\,\mathrm{d}u^{-1} + u\rho u^*, && \theta \mapsto \mathrm{Ad}_u(\theta), \quad \psi \mapsto u\psi, \end{aligned} \tag{24}$$

where $u \in \mathcal{U}(\mathfrak{g})$ and $\psi \in h$. The bosonic and fermionic actions are defined in the same way as in the Connes–Lott prescription: Using the Dixmier trace Tr_ω we define the bosonic action

$$S_{\mathrm{B}}(\nabla) := \min_{j^2 \in \hat{\mathfrak{c}}^2\mathfrak{g} + \pi(\mathcal{J}^2\mathfrak{g})} \mathrm{Tr}_\omega((\theta_0 + j^2)^2 |D|^{-\mathrm{d}}), \tag{25}$$

where $\theta_0 \in \mathcal{H}^2\mathfrak{g}$ is any representative of θ. For the fermionic action we use the scalar product on the Hilbert space:

$$S_{\mathrm{F}}(\psi, \nabla_h) := \langle \psi, \mathrm{i}\nabla_h \psi \rangle_h, \quad \psi \in h. \tag{26}$$

Both S_{B} and S_{F} are invariant under gauge transformations (24).

5. Functions ⊗ matrices

In physical applications one is especially interested in the case that the Lie algebra $\mathfrak{g}$ is the tensor product of the algebra of functions on the space–time manifold X and a matrix Lie algebra $\mathfrak{a}$. We are able to handle this situation. However, it turns out that we must impose restrictions on the matrix Lie algebra. If $\mathfrak{a}$ is semisimple then there are no problems at all. The situation that $\mathfrak{a}$ is Abelian cannot be satisfactorily treated. We are able to deal with L-cycles over the Lie algebra

$$\mathfrak{g} = C^\infty(X) \otimes (\mathfrak{a}' \oplus \mathfrak{a}''), \tag{27}$$

where $C^\infty(X)$ is the algebra of real smooth functions over the (four-dimensional) space–time manifold, $\mathfrak{a}'$ is a semisimple Lie algebra and $\mathfrak{a}''$ an optional Abelian Lie algebra. For $\mathfrak{a}''$ we have to impose constraints on the representations. Remarkably, for the models I considered so far, the u(1)-representations realized in nature are admissible. The Hilbert space is

$$h = L^2(X, S) \otimes \mathbb{C}^F, \tag{28}$$

where $L^2(X, S)$ is the Hilbert space of square integrable sections on the spinor bundle over X. The representation π of $\mathfrak{g}$ on h is given by

$$\pi = \mathrm{id} \otimes \hat{\pi}, \tag{29}$$

where $\hat{\pi}$ is a representation of $\mathfrak{a}' \oplus \mathfrak{a}''$ on $\mathbb{C}^F$. The self-adjoint operator D of the L-cycle is

$$D = \mathsf{D} \otimes 1_F + \gamma^5 \otimes \mathcal{M}, \tag{30}$$

where D and γ^5 are the Dirac operator of the spin connection and the chirality operator on $L^2(X, S)$. Moreover, $\mathcal{M}$ is a symmetrical complex $F \times F$-matrix such that there exists a symmetrical $F \times F$-matrix $\hat{\Gamma}$, fulfilling $\hat{\Gamma}^2 = 1_F$, $\mathcal{M}\hat{\Gamma} = -\hat{\Gamma}\mathcal{M}$ and $\hat{\pi}(a)\hat{\Gamma} = \hat{\Gamma}\hat{\pi}(a)$, for all $a \in \mathfrak{a}$. Then the chirality operator is

$$\Gamma = \gamma^5 \otimes \hat{\Gamma}. \tag{31}$$

As mentioned before, the representation $\hat{\pi}(\mathfrak{a}'')$ is not arbitrary, we have a constraint relation between $\mathcal{M}$ and $\hat{\pi}(\mathfrak{a}'')$, see [26]. Observe that the tuple $(\mathfrak{a}, \mathbb{C}^F, \mathcal{M}, \hat{\pi}, \hat{\Gamma})$ itself forms an L-cycle. In some sense, the L-cycle $(\mathfrak{g}, h, D, \pi, \Gamma)$ is the product of the Dirac K-cycle $(C^\infty(X), L^2(X, S), \mathsf{D}, \gamma^5)$ with the matrix L-cycle $(\mathfrak{a}, \mathbb{C}^F, \mathcal{M}, \hat{\pi}, \hat{\Gamma})$.

One may ask how the spaces $\pi(\Omega^*\mathfrak{g})$, $\pi(\mathcal{J}^*\mathfrak{g})$ and $\Omega^*_D\mathfrak{g}$ depend on the geometric objects of the underlying Dirac K-cycle and the matrix L-cycle. It turns out that $\pi(\Omega^*\mathfrak{g})$, $\pi(\mathcal{J}^*\mathfrak{g})$ and $\Omega^*_D\mathfrak{g}$ can be universally written as a sum of tensor products of differential forms of homogeneous degree (partly coboundaries only) with certain commutators and anticommutators of homogeneous subspaces of $\hat{\pi}(\Omega^*\mathfrak{a})$ and $\hat{\pi}(\mathcal{J}^*\mathfrak{a})$. Thus, if one has complete knowledge of $\hat{\pi}(\Omega^*\mathfrak{a})$ and $\hat{\pi}(\mathcal{J}^*\mathfrak{a})$, then also $\pi(\Omega^*\mathfrak{g})$, $\pi(\mathcal{J}^*\mathfrak{a})$ and $\Omega^*_D\mathfrak{g}$ are known. The formulae of lowest degree read:

$$\begin{aligned}
\pi(\Omega^0\mathfrak{g}) &= \Lambda^0 \otimes (\hat{\pi}(\mathfrak{a}') \oplus \hat{\pi}(\mathfrak{a}'')),\\
\pi(\Omega^1\mathfrak{g}) &= (\Lambda^1 \otimes \hat{\pi}(\mathfrak{a}')) \oplus (B^1 \otimes \hat{\pi}(\mathfrak{a}'')) \oplus (\Lambda^0\gamma^5 \otimes \hat{\pi}(\Omega^1\mathfrak{g})),\\
\pi(\Omega^2\mathfrak{g}) &= (\Lambda^2 \otimes \hat{\pi}(\mathfrak{a}')) \oplus (\Lambda^1\gamma^5 \otimes \hat{\pi}(\Omega^1\mathfrak{a})) \oplus (\Lambda^0 \otimes (\hat{\pi}(\Omega^2\mathfrak{a})\\
&\quad + \{\hat{\pi}(\mathfrak{a}), \hat{\pi}(\mathfrak{a})\})),\\
\pi(\mathcal{J}^0\mathfrak{g}) &= 0, \quad \pi(\mathcal{J}^1\mathfrak{g}) = 0, \quad \pi(\mathcal{J}^2\mathfrak{g}) = \Lambda^0 \otimes (\hat{\pi}(\mathcal{J}^2\mathfrak{a}) + \{\hat{\pi}(\mathfrak{a}), \hat{\pi}(\mathfrak{a})\}),\\
\Omega^0_D\mathfrak{g} &= \pi(\Omega^0\mathfrak{g}), \quad \Omega^1_D\mathfrak{g} = \pi(\Omega^1\mathfrak{g}),\\
\Omega^2_D\mathfrak{g} &= (\Lambda^2 \otimes \hat{\pi}(\mathfrak{a}')) \oplus (\Lambda^1\gamma^5 \otimes \hat{\pi}(\Omega^1\mathfrak{a}))\\
&\quad \oplus (\Lambda^0 \otimes ((\hat{\pi}(\Omega^2\mathfrak{a}) + \{\hat{\pi}(\mathfrak{a}), \hat{\pi}(\mathfrak{a})\}) \operatorname{mod} (\hat{\pi}(\mathcal{J}^2\mathfrak{a}) + \{\hat{\pi}(\mathfrak{a}), \hat{\pi}(\mathfrak{a})\}))).
\end{aligned} \tag{32}$$

Here, Λ^k is the space of k-differential forms, $B^1 = \mathbf{d}\Lambda^0 \subset \Lambda^1$ the space of 1-coboundaries and

$$\{\hat{\pi}(\mathfrak{a}), \hat{\pi}(\mathfrak{a})\} = \left\{ \sum_\alpha \{\hat{\pi}(a_\alpha), \hat{\pi}(\tilde{a}_\alpha)\},\ a_\alpha, \tilde{a}_\alpha \in \mathfrak{a},\ \text{finite sum} \right\}. \tag{33}$$

For higher degrees, the formulae for the matrix part belonging to a fixed space of k-differential forms become more and more complicated. Corresponding formulae in usual NCG formulations are less difficult, because an associative algebra does not care, at which sites in the product $\omega_1 \circ \omega_2 \circ \cdots \circ \omega_n$ one inserts brackets distinguishing commutators and anticommutators. As it can be seen, the Abelian Lie algebra $\mathfrak{a}''$ plays a special role. For instance, if the connection form ρ belongs to $\Omega^1_D\mathfrak{g}$, then the field strength of a u(1)-gauge field is always zero. That u(1)-gauge fields can have a non-vanishing field strength in our theory is due to the extension of $\Omega^1_D\mathfrak{g}$ to $\hat{\mathcal{H}}^1\mathfrak{g}$.

An additional feature of L-cycles over functions ⊗ matrix Lie algebra is the possibility to consider local connections. For local connections, the connection form ρ commutes with functions. Therefore, it has the decomposition

$$\rho \in (\Lambda^1 \otimes \mathbf{r}^0\mathfrak{a}) \oplus (\Lambda^0\gamma^5 \otimes \mathbf{r}^1\mathfrak{a}), \tag{34}$$

where $\mathbf{r}^0\mathfrak{a}$ and $\mathbf{r}^1\mathfrak{a}$ are certain subspaces of $\mathrm{M}_F\mathbb{C}$. The defining equations (19), decomposed according to their differential form degree, yield certain equations for commutators and anticommutators of $\mathbf{r}^0\mathfrak{a}$ and $\mathbf{r}^1\mathfrak{a}$ with $\hat{\pi}(\Omega^*\mathfrak{a})$ and $\hat{\pi}(\mathcal{J}^*\mathfrak{a})$. These equations and $\mathbb{Z}_2$-grading properties and involution identities make it possible to find the space of gauge potentials (34). Moreover, one also gets a decomposition for the ideal $\mathbb{J}^2\mathfrak{g} := \tilde{\mathbf{c}}^2\mathfrak{g}+\pi(\mathcal{J}^2\mathfrak{g})$ commuting with functions, which we need to write down the bosonic action (25):

$$\mathbb{J}^2\mathfrak{g} = (\Lambda^0 \otimes \mathbf{c}^2\mathfrak{a}) \oplus (\Lambda^1\gamma^5 \otimes \mathbf{c}^1\mathfrak{a}) \oplus (\Lambda^2 \otimes (\mathbf{c}^0\mathfrak{a}+\hat{\pi}(\mathcal{J}^2\mathfrak{a}) + \{\hat{\pi}(\mathfrak{a}), \hat{\pi}(\mathfrak{a})\})). \tag{35}$$

Again, one finds certain equations between $\mathbf{c}^i\mathfrak{a}$ and $\hat{\pi}(\Omega^*\mathfrak{a})$ that make it possible to determine $\mathbb{J}^2\mathfrak{g}$. For the computation of the bosonic action one makes use of the fact that in the present situation one can express the Dixmier trace by a combination of the usual trace over the matrix structures (including gamma matrices) and integration over the space–time manifold.

6. Electrodynamics and standard model

One can try to formulate the chiral spinor electrodynamics within our approach. However, since the Lie algebra to use is purely Abelian, there occur certain problems. It is no problem to get the correct fermionic action. In particular, the photon has the usual properties and a non-vanishing classical curvature. Nevertheless, in our approach we get a vanishing curvature and, therefore, no bosonic action.

The reformulation of the standard model [27] is more successful. The L-cycle is the direct transcription of the physical situation. Clearly, the Lie algebra to use is $C^\infty(X) \otimes (\mathrm{su}(3) \oplus \mathrm{su}(2) \oplus \mathrm{u}(1))$. We can formulate the standard model with or without right neutrinos. For a generic mass matrix, the generalized gauge potential ρ contains the usual Yang–Mills fields of the standard model and one complex Higgs doublet. The bosonic Lagrangian includes the Yang–Mills part, the covariant derivative of the Higgs fields and the well-known quartic Higgs potential. Three Higgs components are absorbed by the Higgs mechanism and give mass to the $W^\pm$ and Z bosons. One massive scalar Higgs field survives. In the same way as in the Connes–Lott prescription we obtain tree-level predictions for all bosonic masses. For the simplest scalar product we find in the case that right neutrinos are included

$$m_W = \tfrac{1}{2}m_t, \quad m_Z = m_W/\cos\theta_W, \quad \sin^2\theta_W = \tfrac{3}{8}, \quad m_{\mathrm{H}} = \tfrac{3}{2}m_t. \tag{36}$$

Without right neutrinos, the only modification is $m_{\mathrm{H}} = \sqrt{43/20}m_t$. Here, m_t, m_W, m_Z, m_{H} are the masses of the top quark, the W bosons, the Z boson and the Higgs boson. The photon and the gluons remain massless. The Weinberg angle θ_W coincides with the SU(5)-GUT prediction. Moreover, we get the same coupling constants for the weak and strong interactions. In the Connes–Lott formulation of non-commutative geometry one uses the algebra $\mathcal{A} = C^\infty(X) \otimes (M_3\mathbb{C} \oplus \mathbb{H} \oplus \mathbb{C})$ to derive the standard model, together with a rather complicated representation of $\mathcal{A}$. For the simplest scalar product,[6] the numerical results are [20]

$$\begin{aligned} m_W &= \tfrac{1}{2} m_t, & m_Z &= m_W/\cos\theta_W, \\ \sin^2\theta_W &= \tfrac{12}{29}, & m_H &= \sqrt{\tfrac{69}{28}}\, m_t \approx 1.57\, m_t. \end{aligned} \tag{37}$$

Thus, we see that our predictions do not differ very much from the standard NCG-prescriptions.

7. The flipped SU(5) × U(1)-grand unification model

This section is a summary of our analysis [28] of the flipped SU(5)× U(1)-model. For the classical treatment of that model see [13].

7.1. The matrix L-cycle

The matrix L-cycle is given by the following data: The matrix Lie algebra is $\mathfrak{a} = \mathrm{su}(5)$. Nevertheless, we will obtain an additional u(1)-gauge field and U(1)-gauge transformations due to the extension of $\pi(\Omega^1\mathfrak{g})$ to $\mathcal{H}^1\mathfrak{g}$. Remarkably, the representation of that u(1)-gauge field on the fermionic Hilbert space is unique and realized in nature![7] The internal Hilbert space is $\mathbb{C}^{192}$. This means that we must deal with huge matrices, a problem which should not be underestimated. The strange number $192 = 4 \cdot 48$ arises because there are 48 fermions in nature (including right neutrinos), and we need four copies of them: Two copies because we need particles and antiparticles in one representation (the SU(5) exchanges particles and antiparticles – proton decay!), and an additional doubling to include the essential grading operator. The 48 fermions occur in three generations, each generation contains 16 fermions. These 16 fermions are assigned to the su(5)-representations $\underline{10}$, $\underline{5}^*$, $\underline{1}$. Now, for $a \in \mathrm{su}(5)$ we define the representation $\hat{\pi}$ of the Lie algebra su(5) of our matrix L-cycle in terms of 48×48-block matrices

$$\hat{\pi}(a) = \left(\begin{array}{cc|cc} \hat{A} & 0 & 0 & 0 \\ 0 & \hat{A} & 0 & 0 \\ \hline 0 & 0 & \overline{\hat{A}} & 0 \\ 0 & 0 & 0 & \overline{\hat{A}} \end{array}\right). \tag{38}$$

[6] In the meantime one prefers to use the whole class of compatible scalar product to obtain "fuzzy mass relations", see [1,16,17,23].

[7] This is a purely algebraic result, for which I have no geometric interpretation. I suppose that this has something to do with anomaly freedom of the model.

In terms of the decomposition $\mathbb{C}^{48} = (\underline{10} \oplus \underline{5}^* \oplus \underline{1}) \otimes \mathbb{C}^3$ we have

$$\hat{A} = \operatorname{diag}(\pi_{10}(a) \otimes 1_3, \overline{\pi_5(a)} \otimes 1_3, 0_3). \tag{39}$$

Here, $\pi_5(a) = a$ is the adjoint representation $\underline{24}$ of su(5) and $\pi_{10}(a)$ the embedding of $\underline{24}$ into $\operatorname{End}(\underline{10}) = \underline{1} \oplus \underline{24} \oplus \underline{75}$. The fact that the su(5) representations are tensorized by 1_3 means that the gauge group does not distinguish between the three generations of fermions.

The mass matrix $\mathcal{M}$ of the L-cycle consists of two different contributions. The first one is diagonal and the other one off-diagonal in the sense of the indicated decomposition into two by two blocks in (38):

$$\mathcal{M} = \left(\begin{array}{cc|cc} 0 & \mathcal{M}_i & \mathcal{M}_f & 0 \\ \mathcal{M}_i^* & 0 & 0 & \mathcal{M}_f \\ \hline \mathcal{M}_f^* & 0 & 0 & \overline{\mathcal{M}_i} \\ 0 & \mathcal{M}_f^* & \mathcal{M}_i^{\mathrm{T}} & 0 \end{array}\right). \tag{40}$$

The 48×48-matrix $\mathcal{M}_f = \mathcal{M}_f^{\mathrm{T}}$ is the fermionic mass matrix. A convenient picture is to imagine the two–two structure as the left–right decomposition. Since mass terms exchange left and right fermions, they must stand in the off-diagonal blocks. With this picture in mind it is not difficult to assign the 3×3-fermion mass matrices M_u, M_d, M_e, M_n, M_N to the 16×16-block matrix $\mathcal{M}_f$. Here, M_u is the mass matrix for the (u, c, t)-quark sector, M_d the mass matrix for the (d, s, b)-quark sector and M_e the mass matrix for the (e, μ, τ)-lepton sector. Moreover, M_n and M_N are Dirac and Majorana mass matrices for the $(\nu_e, \nu_\mu, \nu_\tau)$-neutrino sector. These mass matrices include the fermion masses and generalized Kobayashi–Maskawa mixing angles. Mathematically, the sites where these generation matrices occur in $\mathcal{M}_f$ coincide with a combination of the representations $\underline{5}$, $\underline{45}$ and $\underline{50}$ of su(5). The relevant decomposition rules of tensor products are

$$\operatorname{Hom}(\underline{10}^*, \underline{10}) = \underline{5}^* \oplus \underline{45} \oplus \underline{50}, \quad \operatorname{Hom}(\underline{5}, \underline{10}) = \underline{5} \oplus \underline{45}^*, \quad \operatorname{Hom}(\underline{1}, \underline{5}^*) = \underline{5}^*. \tag{41}$$

Let n, n', m' be appropriate elements of $\underline{5}$, $\underline{45}^*$, $\underline{50}$, in this order. Then one has

$$\mathcal{M}_{\mathrm{f}} := \begin{pmatrix} \mathrm{i}\pi_{10,10}(n) \otimes M_d + \mathrm{i}m' \otimes M_N & \mathrm{i}\pi_{10,5}(n) \otimes M_{\tilde{u}} + \mathrm{i}n' \otimes M_{\tilde{n}} & 0 \\ \mathrm{i}\pi_{10,5}(n)^{\mathrm{T}} \otimes M_{\tilde{u}}^{\mathrm{T}} + \mathrm{i}n'^T \otimes M_{\tilde{n}}^T & 0 & \mathrm{i}\pi_{5,1}(n) \otimes M_e \\ 0 & \mathrm{i}\pi_{5,1}(n)^{\mathrm{T}} \otimes M_e^{\mathrm{T}} & 0 \end{pmatrix}, \tag{42}$$

where $\pi_{10,10}(n)$ is the embedding of $n \in \underline{5}$ into $\operatorname{Hom}(\underline{10}^*, \underline{10})$, $\pi_{10,5}(n)$ the embedding of n into $\operatorname{Hom}(\underline{5}, \underline{10})$ and $\pi_{5,1}(n)$ the embedding of n into $\operatorname{Hom}(\underline{1}, \underline{5}^*)$. Moreover,

$$M_{\tilde{u}} = \tfrac{1}{4}(3M_u + M_n), \qquad M_{\tilde{n}} = \tfrac{1}{4}(M_u - M_n). \tag{43}$$

The block diagonal part $\mathcal{M}_i$ of $\mathcal{M}$ couples left–left and right–right sectors. Thus, it has no interpretation in terms of fermion masses. It is responsible for the desired spontaneous symmetry breaking pattern from $\mathrm{su}(5) \oplus \mathrm{u}(1)$ to $\mathrm{su}(3) \oplus \mathrm{su}(2) \oplus \mathrm{u}(1) \oplus \mathrm{u}(1)$, see item (4) at the very beginning. The non-Abelian part of $\mathrm{su}(5)\oplus\mathrm{u}(1)$ commuting with $\mathcal{M}_i$ must coincide with the non-Abelian part of the standard model Lie algebra. In terms of the decomposition

$$\mathrm{su}(5) = \left(\begin{array}{c|c} \mathrm{su}(3) & \cdot \\ \hline \cdot & \mathrm{su}(2) \end{array}\right) \tag{44}$$

we put

$$m = \mathrm{i}\,\mathrm{diag}(-\tfrac{2}{5}, -\tfrac{2}{5}, -\tfrac{2}{5}, \tfrac{3}{5}, \tfrac{3}{5}) \in \mathrm{su}(5)\,. \tag{45}$$

With this notation, the desired symmetry breaking pattern is achieved for

$$\mathcal{M}_i := \mathrm{diag}(\mathrm{i}\pi_{10}(m) \otimes M_{10}, \overline{-\mathrm{i}\pi_5(m) \otimes M_5}, 0_3), \tag{46}$$

where M_{10} and M_5 are arbitrary 3×3-matrices. In contrast to the parameters entering M_f we have no experimental hints how to choose M_{10} and M_5 except that their norm must be very large. Namely, in the flipped SU(5)× U(1)-GUT there occur interactions which lead to proton decay. It turns out that the lifetime predicted for the proton depends on $\mathrm{tr}(M_{10}M_{10}^* + M_5M_5^*)$. The larger the trace (in units of m_t), the larger is the lifetime of the proton. It is essential that the matrices $M_{u,d,e,n,N}$ and $M_{10,5}$ are generically chosen, because otherwise there would be unwanted contributions from the extension (21). Finally, the grading operator is

$$\hat{\Gamma} = \mathrm{diag}(-1_{48}, 1_{48}, 1_{48}, -1_{48}). \tag{47}$$

7.2. *Remarks on the construction*

To this L-cycle we apply our formalism, which performs the following job: First, it extends the matrix $a \in \mathrm{su}(5)$ to a su(5)-gauge field A. This step is obvious, because we have $A \in \pi(\Omega^1\mathfrak{g}) = \Omega^1_D\mathfrak{g}$. Second, a rather long calculation reveals that those local elements of $\mathcal{H}^1\mathfrak{g}$ that are not already contained in $\pi(\Omega^1\mathfrak{g})$ are u(1)-gauge fields A''. The representations π of A and A'' on the fermionic Hilbert space are fixed by the formalism. In the notation of (38) they are given by

$$\begin{aligned} \pi(A) &= \mathrm{diag}(\tilde{A}, \tilde{A}, \gamma_C\bar{\tilde{A}}\gamma_C, \gamma_C\bar{\tilde{A}}\gamma_C), \\ \tilde{A} &= \mathrm{diag}(\pi_{10}(A) \otimes 1_3, \gamma_C\overline{\pi_5(A)}\gamma_C \otimes 1_3, 0_3), \end{aligned} \tag{48}$$

$$\begin{aligned} \pi(A'') &= \mathrm{diag}(\tilde{A}'', \tilde{A}'', \gamma_C\bar{\tilde{A}}''\gamma_C, \gamma_C\bar{\tilde{A}}''\gamma_C), \\ \tilde{A}'' &= \mathrm{diag}(-\tfrac{1}{2}A''1_{10} \otimes 1_3, -\tfrac{3}{2}\gamma_C\overline{A''}\gamma_C 1_5 \otimes 1_3, -\tfrac{5}{2}A'' \otimes 1_3), \end{aligned} \tag{49}$$

where γ_C is the complex conjugation matrix: $\gamma^\mu = \gamma_C\overline{\gamma^\mu}\gamma_C$, $(\gamma_C)^2 = \pm 1_4$. Third, the formalism extends the matrices

m to a $\underline{24}$-Higgs multiplet $\tilde{\Psi} = \Psi + m$,
n to a complex $\underline{5}$-Higgs multiplet $\tilde{\Phi} = \Phi + n$,
n' to a complex $\underline{45}^*$-Higgs multiplet $\tilde{\Upsilon} = \Upsilon + n'$,
m' to a complex $\underline{50}$-Higgs multiplet $\tilde{\Xi} = \Xi + m'$.

This is an immediate consequence of the fact that m, n, n', m' belong to irreducible representations. Thus, the formalism generates the complete bosonic configuration space of the flipped SU(5) × U(1)-model out of the given L-cycle. In total, there are 224 Higgs fields and 25 gauge bosons. The connection form has the structure

$$\rho = \begin{pmatrix} \tilde{\pi}(A+A'') & \gamma^5\tilde{\pi}(\tilde{\Psi}) & \tilde{\pi}(\tilde{\Phi}+\tilde{\Xi}+\tilde{\Upsilon}) & 0 \\ -(\gamma^5\tilde{\pi}(\tilde{\Psi}))^* & \tilde{\pi}(A+A'') & 0 & \gamma^5\tilde{\pi}(\tilde{\Phi}+\tilde{\Xi}+\tilde{\Upsilon}) \\ -(\gamma^5\tilde{\pi}(\tilde{\Phi}+\tilde{\Xi}+\tilde{\Upsilon}))^* & 0 & -\gamma_C\overline{(\tilde{\pi}(A+A''))}\gamma_C & \overline{\gamma^5\tilde{\pi}(\tilde{\Psi})} \\ 0 & -(\gamma^5\tilde{\pi}(\tilde{\Phi}+\tilde{\Xi}+\tilde{\Upsilon}))^* & -(\overline{\gamma^5\tilde{\pi}(\tilde{\Psi})})^* & -\gamma_C\overline{(\tilde{\pi}(A+A''))}\gamma_C \end{pmatrix}. \tag{50}$$

Here, we have denoted by $\tilde{\pi}$ the embeddings (48) and (49) of the gauge fields A and A'', the embedding (42) of the Higgs multiplets $\tilde{\Phi}$, $\tilde{\Upsilon}$ and $\tilde{\Xi}$ and the embedding (46) of the Higgs multiplet $\tilde{\Psi}$ into $M_{48}\mathbb{C}$ each. Thus, Yang–Mills and Higgs fields are treated in a unified way. Since the embeddings (42) and (46) include the matrices $M_{u,d,e,n,N}$ and $M_{10,5}$, the bosonic masses will depend on the fermion masses and the parameters of $M_{10,5}$.

The bosonic Lagrangian contains the usual Yang–Mills Lagrangian, the covariant derivatives of the Higgs fields and the Higgs potential. The Higgs potential is very complicated as a fourth-order polynomial in 224 variables. All gauge invariant combinations of

$$\pi_{10}(\tilde{\Psi}),\, \pi_5(\tilde{\Psi})\,,\, \pi_{10,10}(\tilde{\Phi}),\, \pi_{10,5}(\tilde{\Phi}),\, \pi_{5,1}(\tilde{\Phi}),\, \tilde{\Upsilon},\, \pi_{10,10}(\tilde{\Upsilon}),\, \tilde{\Xi} \tag{51}$$

really do occur. A computation of the minimum of such a monster seems hopeless. However, we do not have to work. The minimum is simply given by

$$\tilde{\Psi} = m, \quad \tilde{\Phi} = n, \quad \tilde{\Upsilon} = n', \quad \tilde{\Xi} = m'. \tag{52}$$

This is a general feature of non-commutative geometry; the Higgs fields occur already in the broken phase. Just to give an impression of the power of our approach we list few examples of occurring contributions to the Higgs potential. Let

$$\begin{aligned} V_1 &= \tilde{\Psi}^2 - \tfrac{1}{5}\mathrm{tr}(\tilde{\Psi}^2)1_5 - \tfrac{1}{5}\mathrm{i}\tilde{\Psi}, \\ V_2 &= (\tilde{\Upsilon}\tilde{\Upsilon}^*)' + \tfrac{8}{3}\mathrm{i}\tilde{\Psi} - \tilde{\Phi}^*\tilde{\Phi} + \tfrac{1}{5}\mathrm{tr}(\tilde{\Phi}^*\tilde{\Phi})1_5, \\ V_3 &= \tilde{\Upsilon}^*\Upsilon - \tfrac{1}{5}\mathrm{tr}(\tilde{\Upsilon}^*\Upsilon)1_5 + 8\mathrm{i}\tilde{\Psi} + 9\tilde{\Phi}^*\Phi - \tfrac{9}{5}\mathrm{tr}(\Phi^*\Phi)1_5, \\ V_4 &= \Upsilon^*\pi_{10,5}(\tilde{\Phi}) + \pi_{10,5}(\tilde{\Phi})^*\Upsilon - 8\mathrm{i}\tilde{\Psi} - 6\Phi^*\Phi + \tfrac{6}{5}\mathrm{tr}(\Phi^*\Phi)1_5, \\ V_5 &= \Upsilon^*\pi_{10,5}(\Phi) - \pi_{10,5}(\Phi)^*\Upsilon, \\ V_6 &= (\tilde{\Xi}\tilde{\Xi}^*)' + \tfrac{1}{3}\mathrm{i}\tilde{\Psi}. \end{aligned} \tag{53}$$

Here, $\mathrm{i}Y'$ denotes the $\underline{24}$-component of the 10×10-matrix $\mathrm{i}Y$. Then,

$$\textstyle\sum_{i,j=1}^{6} \mu_{ij}\mathrm{tr}(V_i V_j) \tag{54}$$

is a typical contribution to the Higgs potential. If one came to the idea to change the relative coefficients a bit, say, to omit the linear terms in V_i, then (52) is no longer the minimum and one has to deal with the monster. At this point at the latest one realizes the advantage that our scheme brings to gauge field theory. The linear terms in (53) arise from the part

$\sigma(\omega^1)$ in Eq. (17) for the differential. They lead to cubic terms in the Higgs potential, which must not be omitted! Principally, we have the freedom to choose the global parameters in the Higgs potential such as μ_{ij} in (54) arbitrarily (but such that the Higgs potential remains positive definite). In the classical construction this freedom exists indeed, and that is the reason why one obtains no predictions for the masses of the Higgs fields. In our approach, also these global parameters are fixed. They are given by traces over certain combinations of the matrices $M_{u,d,e,n,N}$ and $M_{10,5}$. Thus, if we fix the mass matrix $\mathcal{M}$ then all Higgs masses are determined on tree-level.

In the flipped SU(5)× U(1)-model, the Lie subalgebra which leaves the vacuum (52) invariant is $C^\infty(X) \otimes (\mathrm{su}(3)_\mathrm{C} \oplus \mathrm{u}(1)_\mathrm{EM})$. The $\mathrm{su}(3)_\mathrm{C}$ corresponds to the colour symmetry and the $\mathrm{u}(1)_\mathrm{EM}$ to the symmetry generated by the electric charge of the particles. The remaining 16 gauge degrees of freedom, corresponding to

$$C^\infty(X) \otimes ((\mathrm{su}(5) \oplus \mathrm{u}(1))/(\mathrm{su}(3)_\mathrm{C} \oplus \mathrm{u}(1)_\mathrm{EM})), \tag{55}$$

are used to gauge away 16 Higgs fields, 12 of the $\underline{24}$-representation, three of the $\underline{5}$-representation and one of the $\underline{50}$-representation. This in turn gives a mass to the 16 former gauge bosons corresponding to 55. These are the $W^\pm$ and Z bosons, an additional neutral heavy gauge boson Z' and the 12 leptoquarks X and Y (six each). Thus, there remain 208 Higgs fields

$$\psi'_1, \psi'_2, \psi'_3, \psi_0, \psi_1, \ldots, \psi_8, \phi'_0, \phi_1, \ldots, \phi_6, \upsilon'_0, \upsilon_1, \ldots, \upsilon_{89}, \xi_0, \xi_1, \ldots, \xi_{98}, \tag{56}$$

whose masses are obtained by diagonalization of the bilinear terms of the Higgs potential. It is a tedious procedure to select these bilinear terms (without computer algebra it is almost impossible to avoid errors).

7.3. The SU(5)-grand unification model

If we omit ad hoc the u(1)-gauge field A'' and put M_N equal to zero, we can "derive" the SU(5)-grand unification model out of the flipped SU(5) × U(1)-model. This derivation violates the principles of our prescription of non-commutative geometry. However, if we do not perform the extension (21), then the SU(5)-model is obtained from the same L-cycle introduced above, after renaming $M_u \leftrightarrow M_d$, $M_n \mapsto M_e$ and $M_e \mapsto M_\nu$. If one omits the $\underline{5}$-representations and the matrix M_ν then one gets a model without right neutrinos.

7.4. Physical results from the grand unification model

We present the final results (on tree-level) for the flipped SU(5) × U(1)-grand unification model in Table 1. In this table, we denote by m_t and m_b the masses of the top quark and by m_n and m_N the mass scales of the Dirac and Majorana masses for the neutrinos, respectively. The masses in Table 1 are correct for

$$m_n, m_b < m_t \ll \lambda m_t, (\lambda + \check{\lambda}) m_n < M, m_N, \tag{57}$$

Table 1
The particle masses for the SU(5) × U(1)-model

Particle	Mass	Particle	Mass
1. *The completely neutral Higgs fields*			
ϕ_0'	$(0\ldots 1.45)m_t$	ξ_0	$\left(\sqrt{\frac{1}{60}}\cdots\sqrt{\frac{7}{4}}\right)m_N$
υ_0'	λm_t	υ_{45}	$\frac{1}{2}\sqrt{3}\lambda m_t$
ψ_0	$\sqrt{\frac{2}{5}}m_N$	ψ_3'	$\left(0\cdots\frac{1}{12}\sqrt{\frac{11}{3}}\right)\frac{m_N^2}{M}$
2. *The colour-neutral Higgs fields of charge* ∓ 1			
$\frac{1}{\sqrt{2}}(\upsilon_{18}\pm \mathrm{i}\upsilon_{63})$	$\frac{1}{2}\sqrt{3}\lambda m_t$	$\frac{1}{\sqrt{2}}(\psi_1\pm \mathrm{i}\psi_2)$	$\left(0\cdots\frac{1}{12}\sqrt{\frac{11}{3}}\right)\frac{m_N^2}{M}$
3. *The neutral Higgs fields, for* $i=0,\ldots,7$			
ψ_{i+1}	$\left(0\cdots\frac{1}{12}\sqrt{\frac{11}{3}}\right)\frac{m_N^2}{M}$		
υ_{i+1}	$(\lambda\ldots\lambda+\check{\lambda})m_n$	υ_{i+45}	$(\lambda\ldots\lambda+\check{\lambda})m_n$
ξ_{i+32}	$3M$	ξ_{i+81}	$3M$
4. *The Higgs fields of charge* ∓ 1, *for* $i=0,\ldots,7$			
$\frac{1}{\sqrt{2}}(\upsilon_{19+i}\pm \mathrm{i}\upsilon_{64+i})$	$(\lambda\ldots\lambda+\check{\lambda})m_n$	$\frac{1}{\sqrt{2}}(\xi_{25+i}\pm \mathrm{i}\xi_{74+i})$	$3M$
5. *The Higgs fields of charge* $\mp\frac{1}{3}$, *for* $i=0,1,2$ *and* $j=0,\ldots,5$			
$\frac{1}{\sqrt{2}}(\phi_{1+i}\pm \mathrm{i}\phi_{4+i})$	M	$\frac{1}{\sqrt{2}}(\upsilon_{9+i}\pm \mathrm{i}\upsilon_{54+i})$	M
$\frac{1}{\sqrt{2}}(\upsilon_{12+i}\pm \mathrm{i}\upsilon_{57+i})$	M	$\frac{1}{\sqrt{2}}(\upsilon_{39+i}\pm \mathrm{i}\upsilon_{84+i})$	$2M$
$\frac{1}{\sqrt{2}}(\xi_{44+i}\pm \mathrm{i}\upsilon_{93+i})$	M	$\frac{1}{\sqrt{2}}(\xi_{47+i}\pm \mathrm{i}\upsilon_{96+i})$	$2M$
$\frac{1}{\sqrt{2}}(\xi_{19+j}\pm \mathrm{i}\upsilon_{68+j})$	$2M$	$\frac{1}{\sqrt{2}}(\upsilon_{30+j}\pm \mathrm{i}\upsilon_{75+j})$	M
6. *The Higgs fields of charge* $\pm\frac{2}{3}$, *for* $i=0,1,2$ *and* $j=0,\ldots,5$			
$\frac{1}{\sqrt{2}}(\upsilon_{15+i}\pm \mathrm{i}\upsilon_{60+i})$	M	$\frac{1}{\sqrt{2}}(\upsilon_{36+i}\pm \mathrm{i}\upsilon_{81+i})$	$2M$
$\frac{1}{\sqrt{2}}(\upsilon_{42+i}\pm \mathrm{i}\upsilon_{87+i})$	M	$\frac{1}{\sqrt{2}}(\xi_{41+i}\pm \mathrm{i}\upsilon_{90+i})$	M
$\frac{1}{\sqrt{2}}(\xi_{7+j}\pm \mathrm{i}\xi_{56+j})$	$2M$	$\frac{1}{\sqrt{2}}(\xi_{13+j}\pm \mathrm{i}\xi_{62+j})$	$4M$
7. *The Higgs fields of charge* $\mp\frac{4}{3}$, *for* $i=0,1,2$ *and* $j=0,\ldots,5$			
$\frac{1}{\sqrt{2}}(\upsilon_{27+i}\pm \mathrm{i}\upsilon_{72+i})$	M	$\frac{1}{\sqrt{2}}(\xi_{1+j}\pm \mathrm{i}\upsilon_{50+j})$	$2M$
8. *The neutral massive gauge fields*			
Z	$\sqrt{\frac{2}{5}}m_t$	Z'	$\frac{1}{2}\sqrt{\frac{5}{3}}m_N$
9. *The massive gauge fields of charge* ± 1			
$\frac{1}{\sqrt{2}}(W_1\mp \mathrm{i}W_2)$	$\frac{1}{2}m_t$	Weinberg angle: $\sin^2\theta_W=\frac{3}{8}$	
10. *The leptoquarks leading to proton decay, for* $i=0,1,2$			
$\frac{1}{\sqrt{2}}(X_{1+i}\mp \mathrm{i}X_{4+i})$	M	charge: $\mp\frac{1}{3}$	
$\frac{1}{\sqrt{2}}(Y_{1+i}\mp \mathrm{i}Y_{4+i})$	M	charge: $\pm\frac{2}{3}$	

which is physically plausible. The parameter $M \gg m_t$ is the grand unification scale. Moreover, we assume that the Majorana mass of the right neutrinos is of the same order of magnitude as M. The parameters $M, \lambda, \check{\lambda}$ are certain combinations of the unknown parameters of the matrices M_{10} and M_5. For generic matrices M_{10} and M_5, the masses λm_t and $(\lambda \ldots \lambda + \check{\lambda})m_n$ are not significantly smaller than M and m_N. Let us comment on some observations:

(1) There occur three mass scales in the flipped SU(5)× U(1)-model: The mass scale of the fermions determined by m_t, the grand unification scale M and an intermediate scale determined by λm_t and $(\lambda \ldots \lambda + \check{\lambda})m_n$. All particles with fractional-valued electric charge, which therefore lead to proton decay, have a mass of the order M.

(2) There exists precisely one light Higgs field ϕ_0', whose upper bound for the mass is independent of the grand unification matrices M_{10} and M_5. The reason that only an upper bound can be given is the incomplete knowledge of the input parameters. The Higgs field ϕ_0' is a certain linear combination of neutral Higgs fields of the $\underline{5}$-representation and the $\underline{45}^*$-representation. [8] It has precisely the same properties as the standard model Higgs field.

(3) The predictions for the SU(5)-model are qualitatively the same, except that the gauge field Z' and all Higgs fields ξ_i are absent. Moreover, the electric charges of certain Higgs fields are modified.

(4) The standard model is in perfect agreement with experiment. However, our results show that the low energy sector of both the SU(5) × U(1) and SU(5) GUT's is identical with the standard model. This means that it is not possible to decide by means of present energy experiments which of the three models is correct. One essential advantage of the grand unification models is that they explain why proton and electron have up to the sign the same electric charge. On the other hand, the proton is not a stable particle in grand unified models. Concerning this question, the SU(5) × U(1)-model is favoured over the SU(5)-model, because it yields a larger lifetime for the proton [13].

We see that our prescription of non-commutative geometry has the flexibility to describe grand unification models.

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